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Some new highest-wave solutions for deep-water waves of permanent form

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The classical series expansion procedure of Michell is used to calculate some new highest-wave solutions. These solutions are shown to correspond to the types of gravity waves studied recently by Chen & Saffman (1980). Results are presented for wave profiles, phase speeds, and kinetic and potential energies.

1. Introduction

In a recent paper Chen & Saffman (1980) presented results of numerical calculations for periodic, irrotational gravity waves of permanent form. In addition to the classical Stokes wave, which Chen & Saffman call a regular wave, they obtained solutions for new types of gravity waves, which they termed irregular waves. Their numerical procedure involves calculations for a regular wave with an integral number n wavelengths included in the computation region of length L. These waves are called regular waves of class n. At some amplitude, depending on the value of n, they find a bifurcation point in the regular wave solution, and the new solution branch is followed to yield an irregular wave of class n having n crests per wavelength. No irregular waves of class 1 were found. Chen & Saffman presented detailed computations for the irregular waves of classes 2 and 3 for amplitudes up to near the highest wave amplitudes, although they made no attempt to calculate the highest waves.

The purpose of the present paper is to show that the highest wave limits of the new waves calculated by Chen & Saffman may be obtained from the analysis of Michell (1893). By including the fact that the highest wave must have one sharp crest (with a 120° enclosed angle) per wavelength, Michell developed a series solution for the complex velocity in terms of the complex potential, with the series coefficients determined by a set of nonlinear algebraic equations. Although Michell obtained a solution only for the highest Stokes wave (i.e. a regular wave of class 1), we show in the present work that other solutions for the series coefficients exist that yield the limiting wave solutions for the new wave types studied by Chen & Saffman. In particular, we present specific results for irregular waves of classes 2, 3 and 4.

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2. The governing equations

Consider steady, symmetric, periodic waves propagating under the influence of gravity on the free surface of a fluid of infinite depth. Assume that the fluid is inviscid and incompressible and that the motion is two dimensional and irrotational. In addition, assume the existence of a limiting form of these waves in which the fluid speed at the highest crest is zero.

In a reference frame moving with the wave, define a Cartesian co-ordinate system such that the y axis is perpendicular to the undisturbed free surface, directed opposite to the force of gravity, and the x axis is horizontal, directed to the right. Locate the origin of the co-ordinate system at the summit of one of the crests where the fluid speed is zero. Let the wavelength of the wave be λ and the wave speed (i.e. the fluid speed at infinite depth) be c. For this flow we can define a velocity potential ϕ and a stream function ψ in the conventional manner with $\phi = 0$ at the crest, $\psi = 0$ on the surface, and $\psi < 0$ below the surface. The complex position $z = x+iy$ and the complex potential $f = \phi+i\psi$ are analytical functions of each other.

Following Michell (1893), we choose f as the independent variable and the complex velocity $w = (dz/df)^{-1}$ as the dependent variable. The problem is to determine the function $w(f)$ that is analytic in the lower half of the f-plane and that satisfies the following boundary conditions:

$$\frac{\partial}{\partial\phi}|w(f)|^4 - 4g\,\mathrm{Im}\,[w(f)] = 0 \quad \text{on} \quad \psi = 0; \tag{1}$$

$$w(f) \to c \quad \text{as} \quad \psi \to -\infty. \tag{2}$$

Equation (1) represents the ϕ derivative of the Bernoulli equation:

$$(\partial/\partial\phi)\,(q^2+2gy) = 0,$$

where $q^2 = |w|^2$, $\partial y/\partial\phi = q^{-2}\,\partial\phi/\partial y = -|w|^{-2}\,\mathrm{Im}\,(w)$, and g is the acceleration of gravity.

Stokes (1880) showed that for small $|f|$ an approximate solution to (1) with $w(f) = 0$ at $f = 0$ is

$$w(f) = (3ig/2)^{\frac{1}{3}} f^{\frac{1}{3}}, \tag{3}$$

which states that the crest is a corner with an enclosed angle of 120°. Michell (1893) presented a method that incorporates (3) into a computation of the entire flow field for the highest wave.

3. The solution procedure

We now choose units of length and time such that $\lambda = 2\pi$ and $g = 1$. Following Michell (1893), we assume the following expansion:

$$w(f) = c[1-\exp(-if/c)]^{\frac{1}{3}} \sum_{n=0}^{\infty} b_n \exp(-inf/c). \tag{4}$$

The assumption that the waves are symmetric implies that the b_n must be real. Equation (4) satisfies (2) if $b_0 = 1$, and reduces to (3) in the limit of small

$$|f/c-2\pi N|, \quad N = 0, \pm 1, \pm 2, \ldots .$$

Substitution of the series (4) into (1) and equating the coefficients of the individual trigonometric functions to zero gives the following system of nonlinear algebraic equations for the b_n:

$$(3n+2)A_n-(3n+1)A_{n+1}-F_n/c^2 = 0, \tag{5}$$

for $n = 0, 1, 2, \ldots$, and

$$A_n = \frac{1}{2}\sum_{m=0}^{n} B_m B_{n-m} + \sum_{m=1}^{\infty} B_m B_{n+m}, \tag{6}$$

$$F_n = \frac{18\sqrt{3}}{\pi}\sum_{l=0}^{\infty}\frac{(6l+1)\,b_l}{9(2n+1)^2-(6l+1)^2}, \tag{7}$$

$$B_m = \sum_{l=0}^{\infty} b_l b_{m+l}. \tag{8}$$

Apart from notation and choice of units, these are the same relations obtained by Michell (1893).

Michell (1893) found an approximate solution to (5) by assuming the b_n to be of order b_1^n, neglecting all terms of order higher than b_1^3, and solving the resulting 4 nonlinear equations for b_1, b_2, b_3 and c^2. Michell did not specify the method he used to solve the system of nonlinear equations and presented only one solution, which exhibits one crest per wavelength. From Michell's presentation, one might infer that the system has only one solution; however, if we truncate the series (4) at $n = 1$, the first two equations of (5) reduce to, after elimination of c^2,

$$105b_1^4-18b_1^3+153b_1^2-66b_1+2 = 0. \tag{9}$$

This equation has the following two real roots:

$$b_1 = 0{\cdot}0328,\ 0{\cdot}3767.$$

The first root is close to Michell's value $b_1 = 0{\cdot}0397$, whereas the second root is not. Indeed, if we expand (4) in a Fourier Series,

$$w(f) = 1+(b_1-\tfrac{1}{3})\,e^{-if/c}+\left(b_2-\frac{b_1}{3}-\frac{1}{9}\right)e^{-2if/c}+\ldots, \tag{10}$$

we see that the second solution, $b_1 = 0{\cdot}3767$, nearly eliminates the fundamental of this series, meaning that the series could be dominated by a higher harmonic. This is very indicative of the behaviour that Chen & Saffman found for their irregular gravity waves.

Encouraged by these preliminary results, we decided to explore the solutions of (5) more carefully. To solve (5) we truncated the series (4) after N terms, which reduces the system (5) to $N+1$ equations for the $N+1$ unknowns $b_1, b_2, \ldots, b_N$, and c^2. This system of equations was then solved on a computer using Newton–Raphson iteration. Generally, three or four iterations were required to ensure that the left-hand side of (5) was less than a stipulated tolerance of 10^{-8}.

Once the b_n's and c^2 were determined, we numerically calculated the wave profile, the mean kinetic energy per unit area, and the mean potential energy per unit area. We used an 8-point Gauss–Legendre quadrature routine to perform the necessary integrations.

N	c^2	h/L	T	V	Maximum percentage error
10	1·194164	0·141116	0·038336	0·034586	2
20	1·193253	0·141051	0·038272	0·034545	0·7
40	1·193073	0·141052	0·038277	0·034554	0·1
60	1·193062	0·141056	0·038283	0·034560	0·06
80	1·193065	0·141059	0·038286	0·034563	0·05
100	1·193069	0·141060	0·038288	0·034564	0·05
120	1·193072	0·141061	0·038289	0·034566	0·03

TABLE 1. Wave properties of the regular class 1 wave, calculated for different orders N of the series truncation

For the wave profile, x and y were obtained from the following integrals:

$$x(\phi) = \int_0^\phi \frac{\partial x}{\partial \phi'} d\phi', \tag{11}$$

$$y(\phi) = \int_0^\phi \frac{\partial y}{d\phi'} d\phi', \tag{12}$$

in which $\partial x/\partial \phi$ and $\partial y/\partial \phi$ were computed from (4). Both derivatives are singular at $\phi = 0$, behaving as $\phi^{-\frac{1}{3}} + O(\phi^{\frac{2}{3}})$. The integrals were computed by subtracting the singularities out of the integrands.

The mean kinetic energy per unit area was computed from the formula

$$\begin{aligned} T &= -\frac{c}{4\pi}\int_0^{c\pi} (y + \tfrac{1}{2}c^2)\, d\phi \\ &= -\frac{c}{8\pi}\int_0^{c\pi} (c^2 - |w(\phi)|^2)\, d\phi. \end{aligned} \tag{13}$$

The mean potential energy per unit area was computed from the formula

$$\begin{aligned} V &= \frac{1}{2\pi}\int_0^{\pi} y^2\, dx \\ &= \frac{1}{8\pi}\int_0^{c\pi} (c^2 - |w(\phi)|^2)^2 \frac{\partial x}{d\phi}\, d\phi. \end{aligned} \tag{14}$$

Again, (14) was computed by subtracting the singularity at $\phi = 0$ due to $\partial x/\partial \phi$ out of the integrand.

The numerical procedure was programmed in FORTRAN IV and run on the CDC 7600 computer at the Lawrence Berkeley Laboratory.

4. Results and discussion

Regular wave of class 1

As a check on the numerical procedure we first calculated the Stokes wave of maximum height. Michell's values for the b_n's and c^2 were used to begin the calculation. Table 1 shows the resulting values obtained for the wave properties, with the number of terms N in the truncated series taking on various values between 10 and 120. Also, computed

Class	Type	Author	c^2	h/L†	T	V
1	Regular	Present paper	1·1931	0·14106	0·03829	0·03457
1	Regular	Longuet-Higgins & Fox (1978)	1·1931	0·14107	0·03829	0·03457
1	Regular	Cokelet (1977)	1·1928	0·14105	0·03827	0·03457
1	Regular	Chen & Saffman (1980); $b = 0{\cdot}998$‡	1·1932	0·14087	0·03830	0·03457
2	Regular§	Present paper	0·5965	0·07053	0·009572	0·008642
2	Irregular	Present paper	0·5830	0·06907	0·008661	0·007945
2	Irregular	Chen & Saffman (1980); $b = 0{\cdot}98997$	0·5832	0·06862	0·008675	0·007955
3	Regular	Present paper	0·3977	0·04702	0·004254	0·003841
3	Irregular	Present paper	0·3856	0·04575	0·003716	0·003427
3	Irregular	Chen & Saffman (1980); $b = 0{\cdot}99023$	0·3857	0·04545	0·003719	0·003430
4	Regular	Present paper	0·2983	0·03527	0·002393	0·002160
4	Irregular	Present paper	0·2881	0·03417	0·002052	0·001898

† $L = n\lambda$, for regular waves of class n, $L = \lambda$ for irregular waves.

‡ The parameter b varies from zero for an infinitesimal wave to unity for the wave of maximum height.

§ Regular waves of class n are related to regular waves of class 1 by the following relations:

$$c^2 = c_1^2/n,\ h/L = (h/L)_1/n,\ T = T_1/n^2 \quad\text{and}\quad V = V_1/n^2.$$

TABLE 2. Comparisons between different calculations of wave properties

values for $|w|^2$ and y were substituted into the left-hand side of the nondimensional Bernoulli equation $-2y/|w|^2 = 1$ to test the accuracy of the calculations at 100 points equally spaced in ϕ. The maximum percentage departure of the left-hand side of the equation from unity is shown in the last column of table 1. The error decreases with increasing N and the wave properties all appear to converge to values accurate to four or five significant figures.

The height to wavelength ratios h/L presented in table 1 were calculated from the following alternating series, which is derived directly from the Bernoulli equation:

$$\frac{h}{L} = (2)^{\frac{2}{3}} \frac{c^2}{4\pi}\left[\sum_{n=0}^{N}(-1)^n b_n\right]^2. \tag{15}$$

Our computed values for the properties of the regular class 1 wave ($N = 120$) are shown in table 2 to be in good agreement with other recently published values.

The first column of data in table 3 shows the first 25 b_n values obtained for the class 1 wave with truncation at $N = 120$. The coefficients decrease rapidly at first, but then decrease slowly. The slow convergence of the Michell method was noted by Jeffreys (1951), and may reflect some of the mathematical difficulties in the Michell method pointed out by Grant (1973) and Norman (1974). However, for practical purposes, our calculations indicate that profiles and wave properties may be calculated with sufficient accuracy by the Michell method.

Regular and irregular waves of class 2

As an initial approximation to class 2 waves we set $b_1 = \frac{1}{3}$ so that the first harmonic in (10) vanishes. If the initial values of all other coefficients are set equal to zero, the Newton–Raphson procedure yields for $N = 120$ the irregular class 2 wave profile

n	Wave class 1	2	3	4
0	1·00000	1·00000	1·00000	1·00000
1	0·04119	0·34108	0·33558	0·33446
2	0·01252	−0·06531	0·23209	0·22593
3	0·00606	0·07019	−0·11244	0·18454
4	0·00360	−0·02662	0·03914	−0·13983
5	0·00240	0·03059	0·05891	0·01982
6	0·00173	−0·01399	−0·04193	0·04119
7	0·00131	0·01691	0·01400	0·05150
8	0·00102	−0·00832	0·02772	−0·05034
9	0·00083	0·01052	−0·02127	0·00516
10	0·00068	−0·00532	0·00659	0·01691
11	0·00057	0·00703	0·01612	0·02531
12	0·00049	−0·00356	−0·01237	−0·02513
13	0·00042	0·00493	0·00349	0·00159
14	0·00037	−0·00245	0·01042	0·00881
15	0·00033	0·00358	−0·00776	0·01517
16	0·00029	−0·00173	0·00197	−0·01443
17	0·00026	0·00267	0·00719	0·00034
18	0·00023	−0·00124	−0·00510	0·00513
19	0·00021	0·00203	0·00114	0·01005
20	0·00019	−0·00089	0·00517	−0·00895
21	0·00018	0·00158	−0·00346	−0·00014
22	0·00016	−0·00065	0·00067	0·00318
23	0·00015	0·00124	0·00383	0·00707
24	0·00014	−0·00047	−0·00240	−0·00581

TABLE 3. Values for the first 25 coefficients b_n, calculated for the class 1 (regular) wave, and irregular waves of classes 2, 3 and 4

shown in figure 1. Because the irregular wave looks similar to a regular wave that is represented by a truncated series, further calculations were carried out to show that the numerical procedure may be used to obtain a second distinct solution representing the class 2 regular wave. In order to do this, initial b_n values for the class 2 regular wave were calculated directly from the b_n values for the class 1 wave solution. If a sufficient number of these non-zero coefficient values (approximately 100 values for $N = 120$ truncation) are used, then the iteration scheme converges rapidly towards the regular wave solution shown in figure 2. If there are fewer than approximately 80 non-zero initial coefficients the convergence is toward the irregular wave because alternate crests on the initial profile are sufficiently rounded to appeaı similar to the irregular wave.

Table 2 lists the values computed for the highest irregular class 2 wave. These values differ substantially from those for the regular wave, but are close to the values for the largest-amplitude wave calculated by Chen & Saffman. Table 3 lists the first 25 b_n values for the irregular class 2 wave.

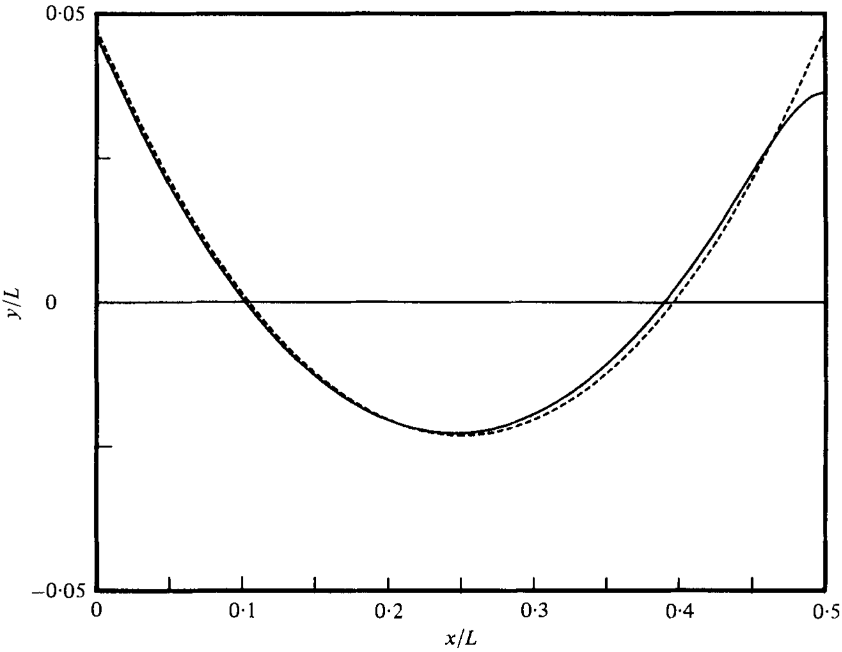

FIGURE 1. Profile of the irregular class 2 wave (solid line), compared with the profile of the regular class 2 wave (dashed line).

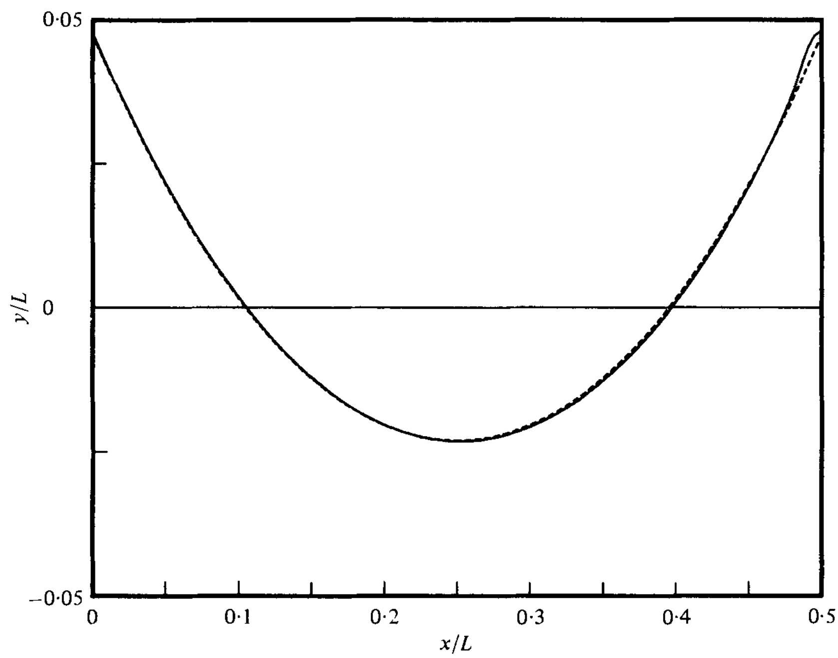

FIGURE 2. Profile of the regular class 2 wave (solid line) obtained by the iteration procedure with $N = 120$, compared with the profile of the regular class 2 wave (dashed line) obtained by directly plotting the class 1 solution with the coordinates reduced by a factor of 2.

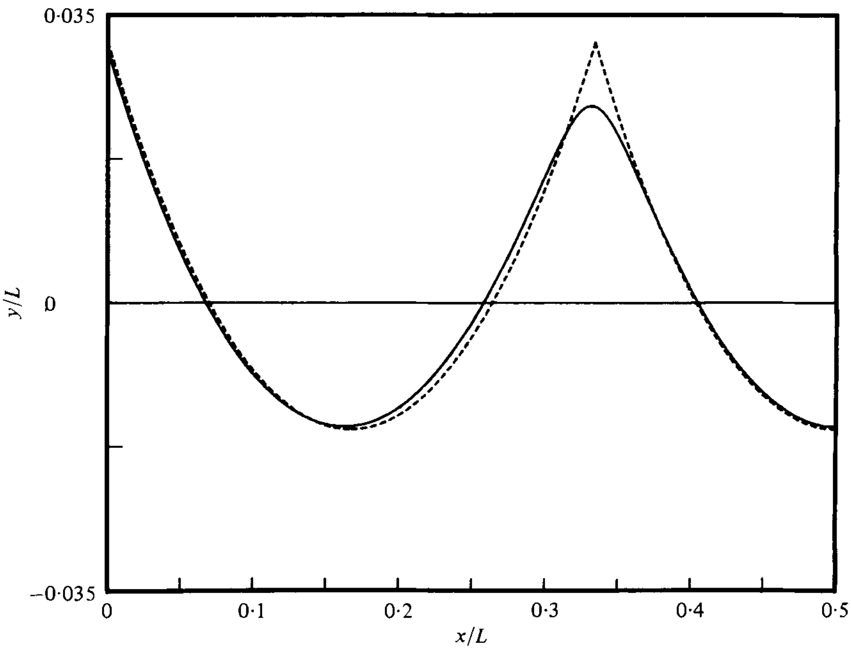

Figure 3. Profile of the irregular class 3 wave (solid line), compared with the profile of the regular class 3 wave (dashed line).

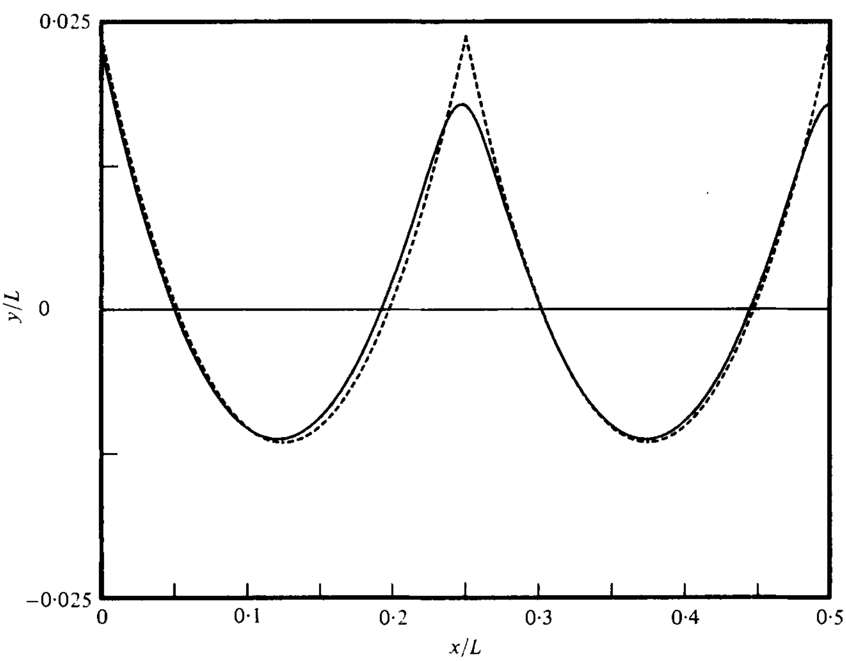

Figure 4. Profile of the irregular class 4 wave (solid line), compared with the profile of the regular class 4 wave (dashed line).

Irregular waves of classes 3 and 4

In order to calculate the higher class waves, initial values for b_1 and b_2 were selected such that the first and second harmonics in (10) vanish for the class 3 wave, and the first, second, and third harmonics vanish for the class 4 wave. The numerical procedure always converged to irregular solutions, and no attempt was made to obtain the regular solutions by iteration. Our iteration procedure yielded only one irregular solution for class 3, and one for class 4, which are plotted in figures 3 and 4, respectively. Chen & Saffman present a second limiting irregular wave of class 3 having two adjacent sharp crests followed by two lower rounded crests. Such a profile is not symmetrical about a sharp crest, and therefore cannot be calculated by the present method. It is expected that class 4 would also have limiting irregular waves which are not symmetrical about a sharp crest and/or contain sharp interior crests which, as discussed previously, are difficult to resolve by the present numerical method.

Computed values for the properties of irregular class 3 and 4 waves are included in table 2. Again, these values are quite distinct from the corresponding values for the regular waves. The first 25 coefficients of the irregular waves are given in table 3. For the higher wave classes the coefficients decrease less rapidly with n. This result is to be expected because the higher numbered coefficients are required to accurately determine the narrower crests occurring in the higher wave classes. Also, note from table 3 that as the class number increases, the coefficients b_1 and b_2 become closer to values $\frac{1}{3}$ and $\frac{2}{9}$, respectively, which are the values which make the first two harmonics in (10) vanish.

5. Conclusions

Michell's (1893) series expansion procedure was shown to yield new solutions, identified as the highest-wave solutions for a new type of (irregular) waves studied by Chen & Saffman (1980). As a check of the procedure, Michell's series expansion, truncated at 120 terms, was shown to yield accurate results when applied to the regular class 1 (Stokes) wave. Wave profiles and values for the phase velocity, height-to-length ratio, kinetic energy per unit area and potential energy per unit area were calculated for irregular waves of classes 2 and 3, yielding good agreement with the largest amplitude waves calculated by Chen & Saffman. Results were also presented for a highest irregular wave of class 4, not considered by Chen & Saffman.

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